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FEBRUARY 1965

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An Integral Equation from Diffraction Theory

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PREPARED UNDER
CONTRACT NO. NONR-285(55)
WITH THE
OFFICE OF NAVAL RESEARCH
NR 062-160

IMM - 337
C. 2

NR 062-160

IMM-NYU 337
February 1965

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AN INTEGRAL EQUATION FROM DIFFRACTION THEORY

A. S. Peters

1. Introduction

The main purpose of this report is to show how to solve the singular integral equation

$$(1.1) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \frac{ke^{i\beta(x-t)}}{x-t} \right\} dt = h(x) + \lambda\Phi(x) \qquad 0 < x$$

where k is an arbitrary constant; α is real; $\beta = \pm\alpha$; and the second integral means the Cauchy principal value. Particular cases of this equation, namely

$$(1.2) \qquad \int_0^\infty \Phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} \pm \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) \qquad 0 < x ,$$

were introduced and solved by D. S. Jones [1], [2]; in connection with a problem in diffraction theory. In [2], Jones shows that if any one of (1.2) is written

$$\int_0^\infty \phi(t) K_j(x,t) dt = h(x) \tag{1.3}$$

then there exists a resolvent kernel $R_j(x,y)$ such that

$$\int_0^\infty R_j(\xi,x) K_j(x,t) dx = \delta(t-\xi) ;$$

and it follows from this that

$$\phi(\xi) = \int_0^\infty R_j(\xi,x) h(x) dx . \tag{1.4}$$

In the proof of (1.4) Jones starts with the exhibition of the function $R_j(x,y)$. The analysis which leads to the choice of $R_j(x,y)$ in the first place is not presented. In a second paper [3] on (1.2), Jones shows that this equation can be reduced to a pair of dual integral equations; which he solves in terms of elementary functions. The solution, however, involves triple integrals and the subsequent analysis which is used in [3] to return to (1.4) is not inconsiderable. In Section 2 we show a more simple analysis of the dual equations.

The dual equations technique is not appropriate for the analysis of

$$(1.5) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} \pm \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) + \lambda\phi(x) \qquad 0 < x .$$

However, in Section 3 we show that the transform

$$\Phi(u) = \int_0^\infty e^{iut}\, \phi(t)dt$$

satisfies a Cauchy singular integral equation of standard type. In Section 4 we point out that (1.5) is a particular case of the more general equation

$$(1.6) \qquad \int_0^\infty K(x-t)\phi(t)dt + \int_0^\infty K_1(x+t)\phi(t)dt = h(x) + \lambda\phi(x)$$

and we show how the solution of this equation can be reduced to the solution of a Hilbert-Riemann problem when $K_1(T)$ is related to $K(T)$ in a special way.

The methods of Sections 3 and 4 are combined in Sections 5 and 6 to show how the solutions of

$$(1.7) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \frac{ke^{-i\alpha(x-t)}}{x-t} \right\} \phi(t)dt = h(x) + \lambda\phi(x)$$

and

$$(1.8) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \frac{ke^{i\alpha(x-t)}}{x-t} \right\} \phi(t)dt = h(x) + \lambda\phi(x)$$

can be found.

2. The Reduction to Dual Equations.

Let us start with

$$(2.1)\qquad \int_0^\infty \phi(t)\left\{\frac{e^{-i\alpha(t-x)}}{t-x}+\frac{e^{-i\alpha(t+x)}}{t+x}\right\}dt = h(x)\qquad x>0$$

where α is real and positive. We suppose that $\phi(t)$ satisfies a uniform Hölder condition when t is positive and finite. This guarantees the existence of the Cauchy principal value of the first integral for each positive and finite value of x. Equation (2.1) is the same as

$$(2.2)\qquad \int_0^\infty \phi_1(t)\left\{\frac{e^{-i(t-x)}}{t-x}+\frac{e^{-i(t+x)}}{t+x}\right\}dt = h_1(x)\qquad x>0$$

where

$$\phi_1(t) = \phi(\tfrac{t}{\alpha})\ ;\quad h_1(x) = h(\tfrac{x}{\alpha})\ .$$

Jones remarked that

$$\int_0^\infty \left\{\frac{e^{-i(t-x)}}{t-x}+\frac{e^{i(t+x)}}{t+x}\right\}\cos\lambda x\,dx = \begin{cases}-\pi\, i\cos\lambda t & 0<\lambda<1\\ \pi\sin\lambda t & 1<\lambda\end{cases}$$

and using this, he reduced (2.2) to the pair of dual equations

$$(2.3)\qquad \int_0^\infty \phi_1(t)\cos\lambda t\,dt = \frac{i}{\pi}\int_0^\infty h_1(x)\cos\lambda x\,dx\qquad 0<\lambda<1$$

$$(2.4) \qquad \int_0^\infty \phi_1(t) \sin \lambda t \, dt = \frac{1}{\pi}\int_0^\infty h_1(x) \cos \lambda x \, dx \qquad 1 < \lambda .$$

In order to solve these, Jones introduced $g_1(\lambda)$ such that

$$(2.5) \qquad \int_0^\infty \phi_1(t) \sin \lambda t \, dt = g_1(\lambda) \qquad 0 < \lambda < 1 .$$

This, with (2.4), implies

$$(2.6) \qquad \frac{\pi}{2} \phi_1(t) = \int_0^1 g_1(\lambda) \sin t\lambda d\lambda + \frac{1}{\pi}\int_1^\infty \sin t\lambda \int_0^\infty h_1(x)\cos \lambda x dx d\lambda$$

After the substitution of (2.6) in (2.3), the use of generalized functions leads to a Cauchy type integral equation which is not difficult to solve for $g_1(\lambda)$; and when $g_1(\lambda)$ is known (2.6) gives $\phi_1(t)$. As was noted in the introduction, the representation for $g_1(t)$ that is obtained in this way is rather complicated, and it seems to require intricate analysis to pass from the form (2.6) to the simpler form (1.4).

Instead of the above method let us follow a method that was developed in [4] by the author. First notice that if $\lambda = 0$ the terms of (2.3) may be infinite. In order to cover this possibility we introduce the generalized function $\delta(\lambda)$ and write (2.3) and (2.4) as

$$(2.7) \qquad \int_0^\infty \phi_1(t) \cos \lambda t dt = \frac{1}{\pi}\int_0^\infty h_1(x) \cos \lambda x dx + c_1\delta(\lambda) \quad 0 \le \lambda < 1$$

$$(2.8) \qquad \int_0^\infty \phi_1(t) \sin \lambda t dt = \frac{1}{\pi} \int_0^\infty h_1(x) \cos \lambda x dx \qquad 1 < \lambda .$$

Now if we multiply each side of (2.7) by $2/\pi \sqrt{r^2 - \lambda^2}$; integrate with respect to λ from 0 to $r < 1$; and use

$$J_0 (rt) = \frac{2}{\pi} \int_0^r \frac{\cos t\lambda d\lambda}{\sqrt{r^2-\lambda^2}}$$

we find that (2.7) becomes

$$(2.9) \qquad \int_0^\infty \phi_1(t) J_0(rt)dt = \frac{i}{\pi} \int_0^\infty h_1(x) J_0(rx)dx + \frac{C}{r} \qquad 0 < r < 1$$

In a similar way, if we use

$$J_0(rt) = \frac{2}{\pi} \int_r^\infty \frac{\sin t\lambda d\lambda}{\sqrt{\lambda^2-r^2}}$$

and

$$Y_0(rx) = - \frac{2}{\pi} \int_r^\infty \frac{\cos x\lambda d\lambda}{\sqrt{\lambda^2-r^2}}$$

we find that (2.8) becomes

$$(2.10) \qquad \int_0^\infty \phi_1(t) J_0(rt)dt = - \frac{1}{\pi} \int_0^\infty h_1(x)Y_0(rx)dx \qquad 1 < r .$$

The results (2.9) and (2.10) show that the zeroth order Bessel transform of $\phi_1(t)$ is

$$\int_0^\infty \phi_1(t)J_0(rt)dt = \begin{cases} \frac{i}{\pi}\int_0^\infty h_1(x)J_0(rx)dx + \frac{C}{r} & 0 < r < 1 \\ \\ \frac{1}{\pi}\int_0^\infty h_1(x)Y_0(rx)dx & 1 < r . \end{cases}$$

The inversion of this transform produces

$$(2.11) \quad \frac{\phi_1(t)}{t} = \frac{i}{\pi}\int_0^1 rJ_0(tr)\int_0^\infty h_1(x)J_0(rx)dxdr + C\int_0^1 J_0(tr)dr$$
$$- \frac{1}{\pi}\int_1^\infty rJ_0(tr)\int_0^\infty h_1(x)Y_0(rx)dxdr$$

for the solution of (2.2).

The representation (2.11) can be simplified by using the formula

$$(2.12) \quad \int^z zC_\mu(kz)C_\mu^*(\ell z)dz = z\frac{[kC_{\mu+1}(kz)C_\mu^*(\ell z) - \ell C_\mu(kz)C_{\mu+1}^*(\ell z)]}{k^2 - \ell^2}$$

in which $C_\mu(T)$ and $C_\mu^*(T)$ are any two cylindrical functions of the same order. [See Watson, Theory of Bessel Functions.] Let us interpret the third integral in (2.11) as

$$- \frac{1}{\pi} \lim_{T \to t + i0} \int_1^\infty \mathcal{R} H_0^{(1)}(Tr)\int_0^\infty h_1(x)Y_0(rx)dxdr$$

where $\mathcal{R}\, H_0^{(1)}(Tr)$ denotes the real part of the Hankel function. Then the use of (2.12) gives

$$\frac{\phi_1(t)}{t} = \frac{i}{\pi}\int_0^\infty h_1(x)\,\frac{[tJ_1(t)J_0(x)-xJ_0(t)J_1(x)]}{t^2-x^2}\,dx + C\int_0^1 J_0(tr)dr$$

$$+\frac{1}{\pi}\int_0^\infty h_1(x)\,\frac{[tJ_1(t)Y_0(x)-xJ_0(t)Y_1(x)]}{t^2-x^2}\,dx$$

which is the same as

$$(2.13)\qquad \phi_1(t) = \frac{i}{\pi}\int_0^\infty h_1(x)\,\frac{[txJ_0(t)H_1^{(2)}(x)-t^2J_1(t)H_0^{(2)}(x)]}{x^2-t^2}\,dx$$

$$+ Ct\int_0^1 J_0(tr)dr.$$

If $C = 0$, (2.13) gives

$$\phi(t) = \phi_1(\alpha t) = \frac{i\alpha}{\pi}\int_0^\infty h(x)\,\frac{[txJ_0(\alpha t)H_1^{(2)}(\alpha x)-t^2J_1(\alpha t)H_0^{(2)}(\alpha x)]}{x^2-t^2}\,dx\ .$$

This solution of (2.1) was presented by Jones in [1], [2]. It shows the meaning of one of the resolvent kernels noted in the introduction, namely,

$$R_1(x,t) = \frac{i\alpha}{\pi}\ \frac{[txJ_0(\alpha t)H_1^{(2)}(\alpha x)-t^2J_1(\alpha t)H_0^{(2)}(\alpha x)]}{x^2-t^2}\ .$$

If we refer to (2.12) again we see that (2.13) can also be

written as

$$(2.14) \qquad \phi_1(t) = \frac{i}{\pi}\int_0^\infty h_1(x) \left\{ t\int_0^1 uH_0^{(2)}(xu)J_0(tu)du + \frac{2i}{\pi}\cdot\frac{t}{x^2-t^2} \right\} dx$$

$$+ C\int_0^t J_0(u)du \ .$$

The result (2.13) was given by Jones in [3]. It must be noted, however, that $\int_0^t J_0(u)du$ satisfies the homogeneous dual equations only when these equations are interpreted from the point of view of summability, or generalized functions, that is to say, if $v = \lambda + i\varepsilon$, then

$$\lim_{\varepsilon \to 0+} \mathcal{R} \int_0^\infty e^{ivt}\int_0^t J_0(u)dudt = 0 \qquad 0 \leq \lambda < 1$$

$$\lim_{\varepsilon \to 0+} \mathcal{I} \int_0^\infty e^{ivt}\int_0^t J_0(u)dudt = 0 \qquad 1 < \lambda$$

but

$$\int_0^\infty \int_0^t J_0(u)du \cos \lambda t dt$$

and

$$\int_0^\infty \int_0^t J_0(u)du \sin \lambda t dt$$

do not exist in the ordinary sense. This implies that $\int_0^t J_0(u)du$

does not satisfy the homogeneous equation corresponding to (2.2), namely,

$$(2.15)\qquad \int_0^\infty \phi_0(t)\left\{\frac{e^{-i(t-x)}}{t-x}+\frac{e^{-i(t+x)}}{t+x}\right\}dt = 0 .$$

In fact, it can be verified that

$$\int_0^\infty\int_0^t J_0(u)du\left\{\frac{e^{-i(t-x)}}{t-x}+\frac{e^{-i(t+x)}}{t+x}\right\} dt = -i\pi .$$

Furthermore, it is not difficult to show that if we set $h_1(x) = i\pi$ in

$$\phi_1(t) = \frac{i}{\pi}\int_0^\infty h_1(x)\left\{t\int_0^1 uH_0^{(2)}(xu)J_0(tu)du + \frac{2i}{\pi}\cdot\frac{t}{x^2-t^2}\right\}dx ,$$

then

$$\phi_1(t) = \int_0^t J_0(T)dT .$$

The equation

$$(2.16)\qquad \int_0^\infty \phi(t)\left\{\frac{e^{-i\alpha(t-x)}}{t-x}-\frac{e^{-i\alpha(t+x)}}{t+x}\right\}dt = h(x) \qquad x > 0 ,$$

where α is real and positive, can be reduced to a pair of dual integral equations by taking the sine transform. This equation is the same as

$$(2.17) \qquad \int_0^\infty \phi_1(t) \left\{ \frac{e^{-i(t-x)}}{t-x} - \frac{e^{-i(t+x)}}{t+x} \right\} dt = h_1(x) \qquad x > 0$$

and if we note that

$$\int_0^\infty \left\{ \frac{e^{-i(t-x)}}{t-x} - \frac{e^{-i(t+x)}}{t+x} \right\} \sin \lambda x dx = \begin{cases} -\pi i \sin \lambda t & 0<\lambda<1 \\ -\pi \cos \lambda t & 1<\lambda \end{cases}$$

we can see that the sine transform of (2.17) yields the equations

$$(2.18) \qquad \int_0^\infty \phi_1(t) \sin \lambda t dt = \frac{i}{\pi} \int_0^\infty h_1(x) \sin \lambda x dx \qquad 0 < \lambda < 1$$

$$(2.19) \qquad \int_0^\infty \phi_1(t) \cos \lambda t dt = - \frac{1}{\pi} \int_0^\infty h_1(x) \sin \lambda x dx \qquad 1 < \lambda .$$

If we regard these equations from the point of view of the theory of generalized functions, then differentiation with respect to λ changes them into

$$(2.20) \qquad \int_0^\infty t\phi_1(t) \cos \lambda t dt = \frac{i}{\pi} \int_0^\infty h_1(x) x \cos \lambda x dx + c_2 \delta(\lambda-1) \qquad 0<\lambda<1$$

$$(2.21) \qquad \int_0^\infty t\phi_1(t) \sin \lambda t dt = \frac{1}{\pi} \int_0^\infty h_1(x) x \cos \lambda x dx + c_2 \delta(\lambda-1) \qquad 1 \leq \lambda .$$

The function $\delta(\lambda-1)$ is introduced because the equations (2.18) and (2.19) show a jump discontinuity at $\lambda = 1$. The last equa-

tions have the same form as (2.7) and (2.8); and hence if we operate on (2.20) and (2.21) as we did on the pair (2.7), (2.8) we find

$$\int_0^\infty t\phi_1(t)J_0(rt)dt = \begin{cases} \frac{i}{\pi}\int_0^\infty xh_1(x)J_0(rx)dx + c_2\delta(r-1) & 0<r\leq 1 \\ -\frac{1}{\pi}\int_0^\infty xh_1(x)Y_0(rx)dx + c_2\delta(r-1) & 1\leq r. \end{cases}$$

The inversion of this yields

$$(2.22) \qquad \phi_1(t) = \frac{i}{\pi}\int_0^1 rJ_0(tr)\int_0^\infty xh_1(x)J_0(rx)dxdr + c_2J_0(t)$$

$$-\frac{1}{\pi}\int_1^\infty rJ_0(tr)\int_0^\infty xh_1(x)Y_0(rx)dxdr$$

for the solution of (2.17). If we integrate in the same way as we did above we get

$$\phi_1(t) = \frac{i}{\pi}\int_0^\infty xh_1(x)\frac{[tJ_1(t)J_0(x) - xJ_0(t)J_1(x)]}{t^2 - x^2}dx + c_2J_0(t)$$

$$+\frac{1}{\pi}\int_0^\infty xh_1(x)\frac{[tJ_1(t)Y_0(x) - xJ_0(t)Y_1(x)]}{t^2 - x^2}dx$$

or

$$(2.23) \qquad \phi_1(t) = \frac{i}{\pi}\int_0^\infty h_1(x)\frac{[x^2J_0(t)H_1^{(2)}(x) - xtJ_1(t)H_0^{(2)}(x)]}{x^2 - t^2}dx$$

$$+ c_2J_0(t) .$$

This can also be written as

$$(2.24) \qquad \phi_1(t) = \frac{i}{\pi}\int_0^\infty h_1(x) \left\{ x\int_0^1 uH_0^{(2)}(xu)J_0(tu)du + \frac{2ix}{\pi(x^2-t^2)} \right\} dx$$

$$+ c_2 J_0(t).$$

The solution (2.24) indicates that $J_0(t)$ may satisfy

$$(2.25) \qquad \int_0^\infty \phi_0(t) \left\{ \frac{e^{-i(t-x)}}{t-x} - \frac{e^{-i(t+x)}}{t+x} \right\} dt = 0 .$$

This is actually the case. In contrast with what we found above for $\int_0^t J_0(u)du$ and the homogeneous equation (2.15), it can be verified that $J_0(t)$ satisfies (2.25) and the homogenous dual equations corresponding to (2.18), (2.19); without invoking summability procedures.

3. Reduction to a Cauchy Integral Equation.

We turn here to the equation

$$(3.1) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \nu \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x)+\lambda\phi(x) \quad 0 < x$$

where α is real; ν is plus or minus one; and $\phi(t)$ is subject to the uniform Hölder condition prescribed in Section 2. We proceed to give a technique which reduces (3.1) to a simple, well-known integral equation of Cauchy type.

We assume that the right hand Fourier transform

$$\Phi(\zeta) = \int_0^\infty e^{i\zeta t}\, \phi(t)dt \qquad \zeta = \xi + i\eta$$

exists almost everywhere for $\mathcal{I}\zeta = 0$. The Hölder continuity assumption is sufficient for the validity of

$$(3.2) \qquad \phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu}\, \Phi(u)du \quad .$$

These assumptions imply that $\Phi(\zeta)$ can have no poles on the real axis. Furthermore, $\Phi(\zeta)$ is analytic for $\mathcal{I}\zeta > 0$ and $\Phi(\zeta) \to 0$ when $|\zeta| \to \infty$ with $0 < \arg \zeta < \pi$. The substitution of (3.2) in (3.1) gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-itu}\, \Phi(u)du \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \nu \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x)+\lambda\phi(x)$$

This is the same as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u) \left\{ -e^{ixu}\int_{-\infty}^{\infty} \frac{e^{i(u-\alpha)T}}{T}\, dT + \nu e^{-ixu}\int_{-\infty}^{\infty} \frac{e^{i(u-\alpha)}}{T}\, dT \right\} du$$

$$= h(x) + \lambda\phi(x)$$

or

$$(3.3) \qquad \frac{1}{2}\int_{-\infty}^{\infty} \Phi(u)[-ie^{ixu} + i\nu e^{-ixu}]\ \mathrm{sgn}(u-\alpha)du = h(x) + \lambda\phi(x)$$

where

$$\text{signum}\ \xi = \mathrm{sgn}\ \xi = \begin{cases} 1 & \xi > 0 \\ -1 & \xi < 0 \end{cases} .$$

Note that the transformation can be reversed from (3.3) to (3.1).

The result of multiplying (3.3) by $e^{i\zeta x}$, $\mathcal{I}\zeta > 0$, and integrating from zero to infinity is

$$(3.4) \qquad \frac{1}{2}\int_{-\infty}^{\infty} \frac{\Phi(u)\ \mathrm{sgn}\ (u-\alpha)du}{u+\zeta} + \frac{\nu}{2}\int_{-\infty}^{\infty} \frac{\Phi(u)\ \mathrm{sgn}\ (u-\alpha)du}{u+\zeta}$$

$$= H(\zeta) + \lambda\ \Phi(\zeta) \qquad\qquad \mathcal{I}\zeta > 0$$

where

$$H(\zeta) = \int_0^{\infty} e^{i\zeta x}h(x)dx \quad .$$

From here on to the end of this section we suppose for convenience that $\alpha > 0$. If we let L denote the path along the real axis from $-\infty$ to α ; and let M denote the path along the real axis from α to $+\infty$ we see from (3.4) that

$$- \frac{1}{2} \int_L \frac{\Phi(u)du}{u+\zeta} + \frac{1}{2} \int_M \frac{\Phi(u)du}{u+\zeta} - \frac{\nu}{2} \int_L \frac{\Phi(u)du}{u-\zeta} + \frac{\nu}{2} \int_M \frac{\Phi(u)du}{u-\zeta}$$

$$= H(\zeta) + \lambda\Phi(\zeta) \ .$$

Let L be hinged at α and then rotated through the upper half plane into coincidence with M . The result of doing this is

$$\int_M \Phi(u) \left[\frac{1}{u+\zeta} + \frac{\nu}{u-\zeta}\right] du - \nu\pi i \ \Phi(\zeta) = H(\zeta) + \lambda \ \Phi(\zeta) \ .$$

We suppose now that each of $H(\xi)$ and $\Phi(\xi)$ is continuous for $\xi > \alpha$. This insures the existence of

$$\lim_{\substack{\zeta \to \xi \\ \xi \to \alpha}} \int_L \Phi(u) \left[\frac{1}{u+\zeta} + \frac{\nu}{u-\zeta}\right] du \ .$$

and it follows that

$$(3.5) \quad \int_\alpha^\infty \Phi(u) \left[\frac{1}{u+\xi} + \frac{\nu}{u-\xi}\right] du = H(\xi) + \lambda \ \Phi(\xi) \qquad \alpha < \xi \ .$$

This equation reduces to well known equations if ν is either 1 or -1 .

If $\nu = 1$ we have

$$\int_{\alpha}^{\infty} \frac{\Phi(u)2udu}{u^2-\xi^2} = H(\xi) + \lambda\, \Phi(\xi) \tag{3.6}$$

or, after an obvious transformation,

$$\int_{0}^{\infty} \frac{\Phi(\sqrt{V+\alpha^2})dV}{V-T} = H(\sqrt{T+\alpha^2}) + \lambda\, \Phi(\sqrt{T+\alpha^2}) .$$

The last equation can be solved in various ways. The solution is well known and for $\lambda \neq \pm \pi i$ it is

$$\Phi(\sqrt{V+\alpha^2}) = -\frac{\lambda H(\sqrt{V+\alpha^2})}{\lambda^2+\pi^2} - \frac{V^{1-\theta}}{(\lambda^2+\pi^2)} \int_{0}^{\infty} \frac{H(\sqrt{T+\alpha^2})dT}{T^{1-\theta}(T-V)} + \frac{C}{V^{\theta}}$$

where θ is defined by

$$\pi \cot \pi\theta = \lambda \qquad\qquad 0 \leq \theta < 1 .$$

Hence

$$\Phi(u) = \frac{-\lambda H(u)}{\lambda^2+\pi^2} - \frac{2(u^2-\alpha^2)^{1-\theta}}{(\lambda^2+\pi^2)} \int_{\alpha}^{\infty} \frac{\xi H(\xi)d\xi}{(\xi^2-\alpha^2)^{1-\theta}(\xi^2-u^2)} \tag{3.7}$$

$$+ \frac{c}{(u^2-\alpha^2)}\, \theta .$$

Similarly if $\nu = -1$ we have

$$(3.8) \qquad 2\int_{\alpha}^{\infty} \frac{\overline{\Phi}(u)du}{u^2-\xi^2} = -\frac{H(\xi)}{\xi} - \frac{\lambda\overline{\Phi}(\xi)}{\xi}$$

and if $\lambda \neq \pm \pi i$ the solution of this is

$$(3.9) \qquad \overline{\Phi}(u) = \frac{-\lambda H(u)}{\lambda^2+\pi^2} + \frac{2u}{(\lambda^2+\pi^2)(u^2-\alpha^2)^w}\int_{\alpha}^{\infty} \frac{(\xi^2-\alpha^2)^w H(\xi)d\xi}{\xi^2-u^2}$$

$$+ \frac{cu}{(u^2-\alpha^2)^w}$$

where w is defined by

$$\pi \cot \pi w = -\lambda \qquad\qquad 0 \leq w < 1 \quad .$$

Let us apply the above method to the equation

$$(3.10) \qquad \int_0^{\infty} \phi(t) \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} + \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = h(x) \quad .$$

[This is the equation we obtain if we replace α in (2.16) by $-|\alpha|$ and $h(x)$ by $-h(x)$.] According to what we have above, the right hand Fourier transform of $\phi(t)$ must satisfy

$$\int_{\alpha}^{\infty} \frac{\overline{\Phi}(u)\cdot 2udu}{u^2-\xi^2} = H(\xi) \quad .$$

As we can see from (3.7), with $\lambda = 0$,

$$(3.11) \qquad \Phi(u) = \frac{-2\sqrt{u^2-\alpha^2}}{\pi^2}\int_\alpha^\infty \frac{\xi H(\xi)d\xi}{\sqrt{\xi^2-\alpha^2}\,(\xi^2-u^2)} + \frac{c}{\sqrt{u^2-\alpha^2}}$$

$$= -\frac{2}{\pi^2}\int_0^\infty h(x)\sqrt{u^2-\alpha^2}\int_\alpha^\infty \frac{\xi e^{i\xi x}d\xi}{\sqrt{\xi^2-\alpha^2}\,(\xi^2-u^2)}\,dx + \frac{c}{\sqrt{u^2-\alpha^2}} \ .$$

In order to show that the inverse of the right hand side of (3.11) can be expressed in a form similar to that shown in (2.24) we will use the facts that

$$\frac{d}{d\xi}\int_0^\alpha \frac{\lambda d\lambda}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}} = \frac{\xi\sqrt{u^2-\alpha^2}}{\sqrt{\xi^2-\alpha^2}\,(\xi^2-u^2)} + \frac{u}{u^2-\xi^2}$$

and

$$\frac{d}{d\xi}\int_0^\xi \frac{\lambda d\lambda}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}} = \frac{u}{u^2-\xi^2} \ .$$

Then

$$\sqrt{u^2-\alpha^2}\int_\alpha^\infty \frac{\xi e^{ix\xi}d\xi}{\sqrt{\xi^2-\alpha^2}\,(\xi^2-u^2)} = \int_\alpha^\infty e^{ix\xi}\frac{d}{d\xi}\int_0^\alpha \frac{\lambda d\lambda\ d\xi}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}}$$

$$+ \int_\alpha^\infty \frac{ue^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= -ix\int_0^\alpha \frac{\lambda}{\sqrt{u^2-\lambda^2}}\int_\lambda^\infty \frac{e^{ix\xi}d\xi d\lambda}{\sqrt{\xi^2-\lambda^2}} - \int_0^\alpha e^{ix\xi}\frac{d}{d\xi}\int_0^\xi \frac{\lambda d\lambda d\xi}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}}$$

$$+ \int_\alpha^\infty \frac{ue^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= \frac{\pi x}{2} \int_0^{\alpha} \frac{\lambda H_0^{(1)}(\lambda x)d\lambda}{\sqrt{u^2-\lambda^2}} + \int_0^{\infty} \frac{e^{ix\xi}u d\xi}{\xi^2-u^2}$$

$$= - \frac{\pi i x}{2} \int_0^{\infty} e^{iut} \int_0^{\alpha} \lambda J_0(\lambda t) H_0^{(1)}(\lambda x) d\lambda dt + \int_0^{\infty} \frac{e^{iut} x}{t^2-x^2} dt \ .$$

It follows that the solution of (3.10) is

$$(3.12) \qquad \phi(t) = - \frac{2}{\pi^2} \int_0^{\infty} h(x) \left\{ - \frac{\pi i x}{2} \int_0^{\alpha} \lambda J_0(\lambda t) H_0^{(1)}(\lambda x) d\lambda + \frac{x}{t^2-x^2} \right\} dx$$

$$+ c_1 J_0(\alpha t)$$

or, by integrating the product of Bessel functions according to (2.12),

$$(3.13) \qquad \phi(t) = \frac{i\alpha}{\pi} \int_0^{\infty} h(x) \frac{[x^2 J_0(\alpha t) H_1^{(1)}(\alpha x) - xtJ_1(\alpha t) H_0^{(1)}(\alpha x)]}{x^2-t^2} dx$$

$$+ c_1 J_0(\alpha t) \ .$$

The equation

$$(3.14) \qquad \int_0^{\infty} \phi(t) \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} - \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = h(x)$$

is the one we obtain if we replace α in (2.1) by $-|\alpha|$. For this case, the transform of the solution, under the assumptions given above, must satisfy

$$2\int_{\alpha}^{\infty} \frac{\Phi(u)du}{u^2-\xi^2} = -\frac{H(\xi)}{\xi} .$$

The solution of this, as we can see from (3.9), is

$$(3.15) \qquad \Phi(u) = \frac{2u}{\pi^2\sqrt{u^2-\alpha^2}} \int_{\alpha}^{\infty} \frac{\sqrt{\xi^2-\alpha^2}\, H(\xi)d\xi}{\xi^2-u^2} + \frac{cu}{\sqrt{u^2-\alpha^2}} .$$

In accordance with our assumption that $\Phi(u) \to 0$ as $|u| \to \infty$, we take $c = 0$, and so

$$(3.16) \quad \Phi(u) = \frac{2u}{\pi^2\sqrt{u^2-\alpha^2}} \int_{\alpha}^{\infty} \frac{\sqrt{\xi^2-\alpha^2}\, H(\xi)d\xi}{\xi^2-u^2}$$

$$= \frac{2}{\pi^2} \int_0^{\infty} h(x) \frac{u}{\sqrt{u^2-\alpha^2}} \int_{\alpha}^{\infty} \frac{e^{ix\xi}\sqrt{\xi^2-\alpha^2}}{\xi^2-u^2}\, d\xi dx .$$

This can be inverted by using the same method that was used to invert (3.11). We note that

$$\frac{d}{du}\int_0^{\alpha} \frac{\lambda d\lambda}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}} = -\frac{u\sqrt{\xi^2-\alpha^2}}{\sqrt{u^2-\alpha^2}\,(\xi^2-u^2)} + \frac{\xi}{\xi^2-u^2}$$

and

$$\frac{d}{du}\int_0^{\xi} \frac{\lambda d\lambda}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}} = \frac{\xi}{\xi^2-u^2} .$$

Then

$$\frac{u}{\sqrt{u^2-\alpha^2}}\int_{\alpha}^{\infty} \frac{e^{ix\xi}\sqrt{\xi^2-\alpha^2}\, d\xi}{\xi^2-u^2} = -\frac{d}{du}\int_{\alpha}^{\infty} e^{ix\xi}\int_0^{\alpha} \frac{\lambda d\lambda d\xi}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}}$$

$$+\int_{\alpha}^{\infty} \frac{\xi e^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= -\frac{d}{du}\int_0^{\alpha}\frac{\lambda}{\sqrt{u^2-\lambda^2}}\int_{\lambda}^{\infty}\frac{e^{ix\xi}d\xi d\lambda}{\sqrt{\xi^2-\lambda^2}} + \frac{d}{du}\int_0^{\alpha} e^{ix\xi}\int_0^{\xi}\frac{\lambda d\lambda d\xi}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}}$$

$$+\int_{\alpha}^{\infty}\frac{\xi e^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= -\frac{\pi i}{2}\frac{d}{du}\int_0^{\alpha}\frac{\lambda H_0^{(1)}(\lambda x)d\lambda}{\sqrt{u^2-\lambda^2}} + \int_0^{\infty}\frac{\xi e^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= -\frac{\pi i}{2}\int_0^{\infty} e^{iut}\, t\int_0^{\alpha}\lambda J_0(\lambda t)H_0^{(1)}(\lambda x)d\lambda dt + \int_0^{\infty}\frac{e^{iut}t dt}{t^2-x^2} .$$

Hence we can see that

$$(3.17)\quad \phi(t) = \frac{2}{\pi^2}\int_0^{\infty}h(x)\left\{-\frac{\pi i t}{2}\int_0^{\alpha}\lambda J_0(\lambda t)H_0^{(1)}(\lambda t)d\lambda + \frac{t}{t^2-x^2}\right\}dx$$

or

$$(3.18)\quad \phi(t) = -\frac{i\alpha}{\pi}\int_0^{\infty}h(x)\frac{[xtJ_0(\alpha t)H_1^{(1)}(\alpha x)-t^2J_1(\alpha t)H_0^{(1)}(\alpha x)]dx}{x^2-t^2}$$

is a solution of (3.14).

The result (3.18) implies that the homogeneous equation

$$(3.19)\qquad \int_0^{\infty}\phi(t)\left\{\frac{e^{i|\alpha|(x+t)}}{x+t} - \frac{e^{-i|\alpha|(x-t)}}{x-t}\right\}dt = 0$$

does not possess a non-trivial solution which satisfies the hypotheses of this section. Is there a solution which does not satisfy those hypotheses? This question prompts us to reexamine the possibility of retaining the term

(3.20) $$\Phi_0(u) = \frac{cu}{\sqrt{u^2-\alpha^2}}$$

in (3.15). If we retain this term we are led to a consideration of a generalized function as a possible solution of (3.19). This is so because

$$\Phi_0(u) = \frac{cu}{\sqrt{u^2-\alpha^2}} = c\left[\frac{u}{\sqrt{u^2-\alpha^2}} - 1\right] + c$$

and inversion gives

$$\phi_0(t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} e^{-iut}\left[\frac{u}{\sqrt{u^2-\alpha^2}} - 1\right] du + \frac{c}{2\pi}\int_{-\infty}^{\infty} e^{-iut} du$$

$$= \frac{c}{\pi i}\int_{-\alpha}^{\alpha} \frac{ue^{-iut}du}{\sqrt{\alpha^2-u^2}} + c\delta(t)$$

(3.21) $$\phi_0(t) = -c\,[\alpha J_1(\alpha t) - \delta(t)] .$$

It can be verified, by substituting $\phi_0(t)$ in (3.19), that (3.21) does in fact satisfy the homogeneous equation (3.19).

It is interesting to observe that solutions of

$$\int_0^\infty \phi(t)\left\{\frac{e^{i\alpha(x+t)}}{x+t} - c\,\frac{e^{-i\alpha(x-t)}}{x-t}\right\} dt = h(x)+\lambda\phi(x)$$

can be found by differentiating and integrating solutions of

$$\int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + c\,\frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) + \lambda\phi(x) \quad .$$

For example, take

$$\int_0^\infty \psi_0(t) \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} + \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0 \quad . \tag{3.22}$$

Assume that an analytic solution exists and deform the path of integration so that it lies above the real axis in the neighborhood of x . Thus

$$-\pi i\,\psi_0(x) + \int_{\frown} \psi_0(t) \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} + \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0$$

and differentiation with respect to x gives

$$-\pi i\,\psi_0'(x) \int_{\frown} \psi_0(t)\,\frac{d}{dx} \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} + \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0 \quad .$$

This is the same as

$$-\pi i\,\psi_0'(x) + \int_{\frown} \psi_0(t)\,\frac{d}{dt} \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} - \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0$$

and after an integration by parts it becomes.

$$-\pi i\,\psi_0'(x) - \psi_0(0)\,\frac{[e^{i|\alpha|x} - e^{-i|\alpha|x}]}{x}$$

$$- \int_{\frown} \psi_0'(t) \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} - \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0$$

if there is no contribution from infinity. If we move the path back into coincidence with the real axis we find

$$(3.23) \qquad -\psi_0(0)\,\frac{[e^{i|\alpha|x}-e^{-i|\alpha|x}]}{x}$$

$$-\int_{\frown} \psi_0'(t)\left\{\left[\frac{e^{i|\alpha|(x+t)}}{x+t}-\frac{e^{-i|\alpha|(x-t)}}{x-t}\right]\right\}dt=0$$

Now we know that $\psi_0(t) = J_0(\alpha t)$ satisfies (3.22). Substituting this in (3.23) gives

$$-\frac{[e^{i|\alpha|x}-e^{-i|\alpha|x}]}{x}+\alpha\int_0^\infty J_1(\alpha t)\left\{\frac{e^{i|\alpha|(x+t)}}{x+t}-\frac{e^{-i|\alpha|(x-t)}}{x-t}\right\}dt = 0$$

which can be written

$$\int_0^\infty [\alpha J_1(\alpha t)-\delta(t)]\left\{\frac{e^{i|\alpha|(x+t)}}{x+t}-\frac{e^{-i|\alpha|(x-t)}}{x-t}\right\}dt = 0 \quad .$$

This again justifies the retention of the term (3.20).

4. Reduction to a Hilbert-Riemann Boundary Value Problem.

The equation

$$(4.1) \quad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \nu \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) + \lambda\phi(x) \qquad x > 0 ,$$

where ν is plus or minus one, is a particular case of the more general equation

$$(4.2) \qquad \int_0^\infty [K(x-t) \pm K_1(x+t)] \, \phi(t)dt = h(x) + \lambda\phi(x) \qquad x > 0$$

where we are given $K_1(T) = K(-T)$, $T > 0$, and we define $K_1(T)$ for negative values of T so that $K_1(T) = K(-T)$ holds for all real T . We proceed to show that if the transform $\underline{K}(u)$ exists almost everywhere for u real, then (4.2) can be reduced to the solution of the Hilbert-Riemann problem defined below.

If the Fourier transform of $K(T)$ is

$$\underline{K}(u) = \int_{-\infty}^{\infty} e^{iuT} K(T)dT ,$$

the Fourier transform of $K_1(T)$ is

$$\underline{K}_1(u) = \int_{-\infty}^{\infty} e^{iuT} K_1(T) = \int_{-\infty}^{\infty} e^{iuT} K(-T)dT = \int_{-\infty}^{\infty} e^{-iuT} K(T)dT = \underline{K}(-u)$$

Let the right hand transform of $\phi(t)$ be

$$\underline{\Phi}(\zeta) = \int_0^\infty e^{i\zeta t}\phi(t)dt \qquad \mathcal{I}\zeta > 0$$

and let

$$\underline{\Phi}(\xi) = \lim_{n \to 0+} \underline{\Phi}(\xi + in) = \lim_{n \to 0+} \underline{\Phi}(\zeta)$$

$$\underline{\Phi}(-\xi) = \lim_{n \to 0-} \underline{\Phi}(-\xi - in) = \lim_{n \to 0+} \underline{\Phi}(-\xi + in)$$

$$= \lim_{n \to 0+} \underline{\Phi}(-\overline{\zeta}) = \underline{\Phi}(\zeta e^{i\pi}) \ .$$

From the standpoint of generalized functions we have

$$(4.3) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixu} [\underline{K}(u) \ \underline{\Phi}(u) \pm \underline{K}(-u) \ \underline{\Phi}(-u)]du = h(x) + \lambda\phi(x) \ .$$

If we multiply this by $e^{i\zeta x}, \mathcal{I}\zeta > 0$, and integrate from zero to infinity we get

$$(4.4) \qquad \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[\underline{K}(u) \ \underline{\Phi}(u) \pm \underline{K}(-u) \ \underline{\Phi}(-u)]}{u - \zeta} du = H(\zeta) + \lambda \ \underline{\Phi}(\zeta)$$

$$\mathcal{I}\zeta > 0 \ .$$

Also if we multiply (4.3) by $e^{-i\zeta x}$, $\mathcal{I}\zeta < 0$, and integrate from zero to infinity we get

$$(4.5) \qquad \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[\underline{K}(u) \ \underline{\Phi}(u) \pm \underline{K}(-u) \ \underline{\Phi}(-u)]}{u + \zeta} du = H(-\zeta) + \lambda \underline{\Phi}(-\zeta)$$

$$\mathcal{I}\zeta < 0$$

which is the same as

$$(4.6) \quad -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[\overline{\underline{K}}(-u)\ \overline{\underline{\Phi}}\ (-u) \pm \overline{\underline{K}}(u)\ \overline{\underline{\Phi}}\ (u)]}{u - \zeta}\, du = H(-\zeta) + \lambda\ \underline{\Phi}\ (-\zeta)$$

$$\Im\zeta < 0$$

and we can write this in the form

$$(4.7) \quad -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[\overline{\underline{K}}(-u)\ \overline{\underline{\Phi}}\ (-u) \pm \overline{\underline{K}}(u)\ \overline{\underline{\Phi}}\ (u)]}{u - \overline{\zeta}}\, du = H(-\overline{\zeta}) + \lambda\ \underline{\Phi}\ (-\overline{\zeta})$$

$$\Im\zeta > 0 .$$

If we add (4.7) to (4.4) when the upper sign is in force; and subtract (4.7) from (4.4) when the lower sign holds we find

$$(4.8) \qquad \frac{1}{2\pi i}\int_{-\infty}^{\infty} [\overline{\underline{K}}(u)\ \overline{\underline{\Phi}}(u) \pm \overline{\underline{K}}(-u)\ \overline{\underline{\Phi}}(-u)]\left(\frac{1}{u-\zeta} - \frac{1}{u-\overline{\zeta}}\right) du$$

$$= H(\zeta) \pm H(-\overline{\zeta}) + \lambda[\ \overline{\underline{\Phi}}(\zeta) \pm \overline{\underline{\Phi}}(-\overline{\zeta})] .$$

$$\Im\zeta > 0 .$$

Let us take the limit of (4.8) as $\zeta \rightarrow |\xi|$; and under the supposition that on any finite interval

$$\overline{\underline{K}}(u)\ \overline{\underline{\Phi}}(u) \pm \overline{\underline{K}}(-u)\ \overline{\underline{\Phi}}(-u)$$

is integrable in the Riemann sense or the Cauchy-Riemann sense if

there is a simple pole on the path of integration. The result is that

$$(4.9) \quad \overline{K}(\xi)\ \overline{\Phi}(\xi) \pm \overline{K}(-\xi)\ \overline{\Phi}(-\xi) = H(\xi) \pm H(-\xi) + \lambda[\overline{\Phi}(\xi) \pm \overline{\Phi}(-\xi)]$$

or

$$(4.10) \quad [\overline{K}(\xi)-\lambda]\ \overline{\Phi}(\xi) \pm [\overline{K}(-\xi)-\lambda]\ \overline{\Phi}(\xi e^{i\pi}) = H(\xi) \pm H(\xi e^{i\pi})$$

must hold for almost all values of $\xi \geq 0$. With respect to the z-plane

$$\xi^2 = z = x + iy$$

and with the notation

$$\overline{\Phi}(\sqrt{x}) = F(x) \ ; \ \overline{K}(\sqrt{x}) = Q(x) \ ; \ H(\sqrt{x}) = I(x) \ ;$$

the equation (4.10) is

$$(4.11) \quad [\ Q(x)-\lambda]F(x) \pm [\ Q\ (xe^{2\pi i})-\lambda]F(xe^{2\pi i}) = I(x) \pm I(xe^{2\pi i})$$

where $x \geq 0$. Equation (4.11) is the barrier equation for a Hilbert-Riemann problem. The problem is to find a function $F(z)$ analytic in the finite z-plane cut along the positive real axis and such that (4.11) is satisfied. Any singularities that we may be willing to admit for $F(z)$ at infinity or at the banks of the cut must of course be consistent with the existence of the integral (4.4) and must allow a reversal of the transformation of (4.2) into (4.4).

We have shown how the solution of (4.2) can be reduced to the

solution of the problem posed by the barrier equation (4.11). Since (4.11) can be solved by a well-known procedure, given for example in [5], we do not repeat it here.

5. Another Generalization

The equation

$$(5.1)\qquad \int_0^\infty \phi(t)\left\{\frac{e^{i\alpha(x+t)}}{x+t} + \nu\,\frac{e^{-i\alpha(x-t)}}{x-t}\right\} dt = h(x) + \lambda\phi(x) \qquad x > 0,$$

where $\nu = \pm 1$, can also be regarded as a particular case of the equation

$$(5.2)\qquad \int_0^\infty \phi(t)\left\{\frac{e^{i\alpha(x+t)}}{x+t} + k\,\frac{e^{-i\alpha(x-t)}}{x-t}\right\} dt = h(x) + \lambda\phi(x) \qquad x > 0,$$

where k is an arbitrary constant. The solution of (5.2) can be reduced to the solution of a Hilbert-Riemann problem by using the methods of Sections 3 and 4. We outline the reduction below.

We assume that $\phi(t)$ in (5.2) and its transform

$$\Phi(\xi) = \int_0^\infty e^{i\xi t}\phi(t)\,dt$$

satisfy the conditions set in Section 3. If we take $\alpha > 0$ for convenience, then we find, just as in Section 3, that $\Phi(u)$ must satisfy

$$(5.3)\qquad \int_\alpha^\infty \Phi(u)\left[\frac{1}{u+\xi} + \frac{k}{u-\xi}\right] du = H(\xi) + \lambda\,\Phi(\xi) \qquad \underline{0 < \alpha < \xi}$$

or

$$(5.4)\qquad \int_1^\infty \Phi_1(u)\left[\frac{1}{u+\xi} + \frac{k}{u-\xi}\right] du = H_1(\xi) + \lambda\,\Phi_1(\xi) \qquad 1 < \xi$$

where

$$\Phi_1(u) = \Phi(\alpha u) \; ; \; H_1(\xi) = H(\alpha\xi) \; .$$

Equation (5.4) is an equation of Wiener-Hopf type which can be solved by using Mellin transforms.

We assume that the Mellin transform

$$(5.5) \qquad \Psi(s) = \int_1^\infty u^{s-1} \Phi_1(u) du$$

exists for all s such that $Rs = a$; $0 < a < 1$ and that

$$(5.6) \qquad \Phi_1(u) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} u^{-s} \Psi(s) ds \; .$$

the transform for $\Psi(s)$ is analytic in the left half plane $\mathcal{R} s < a$ and $\Psi(s) \to 0$ as $|s-a| \to \infty$, $\frac{\pi}{2} < \arg(s-a) < \frac{3\pi}{2}$. The substitution of (5.6) in (5.4) gives

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Psi(s) \int_0^\infty u^{-s} \left[\frac{1}{u+\xi} + \frac{k}{u-\xi} \right] du ds = H_1(\xi) + \lambda \Phi_1(\xi)$$

or

$$(5.7) \quad \frac{1}{2i} \int_{a-i\infty}^{a+i\infty} \xi^{-s} \left[\frac{1 + k \cos \pi s}{\sin \pi s} \right] \Psi(s) ds = H_1(\xi) + \lambda \Phi_1(\xi) \; .$$

The result of multiplying (5.7) by ξ^{z-1} , $\mathcal{R} z < a$, and integrating from one to infinity is

$$(5.8) \qquad \frac{1}{2i} \int_{a-i\infty}^{a+i\infty} \frac{1}{s-z} \left[\frac{1 + k \cos \pi s}{\sin \pi s} \right] \Psi(s) ds = \mathcal{H}(z) + \lambda \Psi(z)$$

$$Rz < a$$

where

$$\mathcal{H}(z) = \int_1^\infty \xi^{z-1} H_1(\xi)d\xi$$

is supposed to exist for $Rz = a$.

The path of integration in (5.8) is parallel to the imaginary axis in the s-plane and we denote the path of Γ . Let σ be a point on Γ and let $z \to \sigma$. The limit process yields

$$(5.9) \quad \frac{\pi}{2}\left[\frac{1 + k\cos\pi\sigma}{\sin\pi\sigma}\right]\Psi(\sigma) + \frac{1}{2i}\int_\Gamma \frac{1}{s-\sigma}\left[\frac{1 + k\cos\pi s}{\sin\pi s}\right]\Psi(s)ds$$

$$= \mathcal{H}(\sigma) + \lambda\Psi(\sigma)$$

or

$$(5.10) \quad \pi\left[\frac{1+k\cos\pi\sigma}{\sin\pi\sigma}\right]\Psi(\sigma) + \lim_{z\to\sigma+0}\frac{1}{2i}\int_\Gamma \frac{1}{s-z}\left[\frac{1 + k\cos\pi s}{\sin\pi s}\right]\Psi(s)ds$$

$$= \mathcal{H}(\sigma) + \lambda\Psi(\sigma) .$$

That is, if $F(z)$ is the sectionally analytic function defined by

$$F(z) = \begin{cases} \Psi(z) & \text{analytic for } Rz < a \\ \dfrac{1}{2i}\displaystyle\int_\Gamma \frac{1}{s-z}\left[\frac{1 + k\cos\pi s}{\sin\pi s}\right]\Psi(s)ds & \text{analytic for } Rz > a , \end{cases}$$

then $F(z)$ must satisfy the Hilbert-Riemann problem posed by the barrier equation

$$\left[\frac{\pi (1 + k \cos \pi \sigma)}{\sin \pi\sigma} - \lambda \right] F(\sigma - 0) + F(\sigma + 0) = \mathcal{H}(\sigma) \tag{5.12}$$

which can be solved by known methods. Once $F(z)$ is known, $\Psi(s)$ can be found from (5.11) and it follows by inversion that the solution of (5.2) is

$$\Phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{u}{a}\right)^{-s} \Psi(s) ds du .$$

6. A Related Equation

In this section we show that the solution of the equation

$$(6.1) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{-i\alpha(x+t)}}{x+t} + \frac{ke^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) + \lambda\phi(x) \qquad x > 0$$

can be reduced to the solution of an equation already considered.

We assume that $\phi(t)$ and

$$\Phi(\zeta) = \int_0^\infty e^{i\zeta t}\, \phi(t)dt$$

satisfy the conditions prescribed in Section 3. By the method of that section we are led to

$$(6.2) \qquad \frac{1}{2} \int_{-\infty}^{\infty} \frac{\Phi(u)\ \operatorname{sgn}\ (u+\alpha)}{u+\zeta}\, du + \frac{k}{2} \int_{-\infty}^{\infty} \frac{\Phi(u)\ \operatorname{sgn}\ (u-\alpha)du}{u-\zeta}$$

$$= H(\zeta) + \lambda\, \Phi(\zeta) \qquad \mathscr{I}\zeta > 0\ .$$

Suppose that $\alpha > 0$. The transform $\Phi(\zeta)$ is analytic in $\mathscr{I}\zeta > 0$ and $\Phi(\zeta) \longrightarrow 0$ as $|\zeta| \longrightarrow \infty$, $0 \arg \zeta < \pi$. Hence an application of the Cauchy integral formula to (6.2) produces

$$(6.3) \qquad - \int_{-\pi}^{-\alpha} \frac{\Phi(u)du}{u+\zeta} + k \int_{\alpha}^{\infty} \frac{\Phi(u)du}{u-\zeta} - k\pi i\, \Phi(\zeta) = H(\zeta) + \lambda\, \Phi(\zeta)$$

or

$$(6.4)\qquad \int_{\alpha}^{\infty} \frac{[\underline{\Phi}(ue^{i\pi}) + k\,\underline{\Phi}(u)]}{u-\zeta}\,du - k\pi i\,\underline{\Phi}(\zeta) = H(\zeta) + \lambda\,\underline{\Phi}(\zeta)$$

$$\mathscr{I}\,\zeta > 0\,.$$

If we let ζ in (6.4) approach ξ, $\xi > \alpha$, and then let ζ approach $\xi e^{i\pi}$ we obtain

$$(6.5)\qquad \int_{\alpha}^{\infty} \frac{[\underline{\Phi}(ue^{i\pi}) + k\,\underline{\Phi}(u)]}{u-\xi}\,du + \pi i\,\underline{\Phi}(\xi e^{i\pi}) = H(\xi) + \lambda\,\underline{\Phi}(\xi)$$

$$(6.6)\qquad \int_{\alpha}^{\infty} \frac{[\underline{\Phi}(ue^{i\pi}) + k\,\underline{\Phi}(u)]}{u+\xi}\,du - k\pi i\,\underline{\Phi}(\xi e^{i\pi}) = H(\xi e^{i\pi}) + \lambda\,\underline{\Phi}(\xi e^{i\pi})\,.$$

If we multiply equation (6.5) by k and add the resulting equation to (6.6) we get

$$(6.7)\qquad k\int_{\alpha}^{\infty} \frac{[\underline{\Phi}(ue^{i\pi}) + k\,\underline{\Phi}(u)]}{u-\xi}\,du + \int_{\alpha}^{\infty} \frac{[\underline{\Phi}(ue^{i\pi}) + k\,\underline{\Phi}(u)]}{u+\xi}\,du$$

$$= kH(\xi) + H(\xi e^{i\pi}) + \lambda[k\,\underline{\Phi}(\xi) + \underline{\Phi}(\xi e^{i\pi})]\,.$$

If we introduce

$$(6.8)\qquad k\,\underline{\Phi}(u) + \underline{\Phi}(ue^{i\pi}) = \Omega(u)$$

equation (6.7) becomes

$$(6.9) \qquad \int_{\alpha}^{\infty} \Omega(u) \left[\frac{1}{u+\xi} + \frac{k}{u-\xi}\right] du$$

$$= kH(\xi) + H(\xi e^{i\pi}) + \lambda\Omega(\xi) \ .$$

This equation was discussed in Section 5 and it was shown there how it could be solved by reducing it to a barrier equation. Once $\Omega(u)$ has been found, the transform $\underline{\overline{\Phi}}(u)$ can be determined by solving the Hilbert-Riemann problem posed by the barrier equation (6.8).

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Security Classification

DOCUMENT CONTROL DATA - R&D

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1. ORIGINATING ACTIVITY (Corporate author) Courant Institute of Mathematical Sciences New York University	2a. REPORT SECURITY CLASSIFICATION Not classified
	2b. GROUP None

3. REPORT TITLE

An Integral Equation from Diffraction Theory

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Technical Report February 1965

5. AUTHOR(S) (Last name, first name, initial)

Peters, Arthur S.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
February 1965	38	5

8a. CONTRACT OR GRANT NO. Nonr-285(55)	9a. ORIGINATOR'S REPORT NUMBER(S) IMM-NYU 337
b. PROJECT NO. NR 062-160	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) None
d	

10. AVAILABILITY/LIMITATION NOTICES

All distribution to be controlled

11. SUPPLEMENTARY NOTES None	12. SPONSORING MILITARY ACTIVITY U.S. Navy, Office of Naval Research 207 West 24th St., New York, N. Y.

13. ABSTRACT

This report presents several methods for solving

$$\text{(A)} \quad \int_0^\infty \phi(t)\left\{\frac{e^{i\alpha(x+t)}}{x+t} + \frac{ke^{i\beta(x-t)}}{x-t}\right\}dt = h(x) + \lambda\phi(x) \qquad 0 < x$$

where k; λ are arbitrary constants; α is real and $\beta = \pm\alpha$. Equation (A) is a generalization of an equation which arises in diffraction theory.

The more general equation

$$\text{(B)} \quad \int_0^\infty K(x-t)\phi(t)dt + \int_0^\infty K_1(x+t)\phi(t)dt = h(x) + \lambda\phi(x) \qquad 0 < x$$

is also studied; and it is shown how the solution of (B) can be reduced to the solution of a Hilbert-Riemann problem when $K(\tau)$ is related to $K_1(\tau)$ in a special way.

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